A note on the superintegrability of the Toda lattice[*]

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Abstract

The superintegrability of the non-periodic Toda lattice is explained in the framework of systems written in action-angles coordinates. Moreover, a simpler form of the first integrals is given.

1 Background results

The non-periodic Toda lattice is a celebrated completely integrable system formed by n particles (with coordinates x_i and momenta p_i) moving on a line according to the Hamilton equations

$$
\begin{array}{llll}
\dot{x}_1 &=& p_1 & \dot{p}_1 &=& -\mathrm{e}^{\,x_1-x_2} \\
\dot{x}_2 &=& p_2 & \dot{p}_2 &=& \mathrm{e}^{\,x_1-x_2} - \mathrm{e}^{\,x_2-x_3} \\
&\vdots& & &\vdots& \\
\dot{x}_i &=& p_i & \dot{p}_i &=& \mathrm{e}^{\,x_{i-1}-x_i} - \mathrm{e}^{\,x_i-x_{i+1}} \\
&\vdots& & &\vdots& \\
\dot{x}_n &=& p_n & \dot{p}_n &=& \mathrm{e}^{\,x_{n-1}-x_n}
\end{array}
\tag{1}
$$

the Hamiltonian of the system is

$$
H(q_1, .., q_n, p_1, .., p_n) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{\,q_i-q_{i+1}} .
$$

Introducing the Lax pair

$$
L = \frac{1}{2}\begin{pmatrix}
-p_1 & \mathrm{e}^{\frac{x_1-x_2}{2}} & 0 & \cdots & & 0 \\
\mathrm{e}^{\frac{x_1-x_2}{2}} & -p_2 & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & & 0 \\
\vdots & \ddots & \ddots & \ddots & & \mathrm{e}^{\frac{x_{n-1}-x_n}{2}} \\
0 & \cdots & 0 & \mathrm{e}^{\frac{x_{n-1}-x_n}{2}} & & -p_n
\end{pmatrix}
\tag{2}
$$

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$$
B = \frac{1}{2}
\begin{pmatrix}
0 & -\mathrm{e}^{\frac{x_1-x_2}{2}} & 0 & \cdots & & 0 \\
\mathrm{e}^{\frac{x_1-x_2}{2}} & 0 & \ddots & & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & & 0 \\
\vdots & \ddots & \ddots & \ddots & & -\mathrm{e}^{\frac{x_{n-1}-x_n}{2}} \\
0 & \cdots & 0 & \mathrm{e}^{\frac{x_{n-1}-x_n}{2}} & & 0
\end{pmatrix}
$$

the Hamilton equations (1) imply the Lax equation $\dot{L} = [L, B]$, thus the n eigenvalues λ_i of L are first integrals. Since one can show that the functions λ_i are independent and mutually in involution, the Toda lattice is completely integrable. Other equivalent sets of first integrals in involution are given by the trace of the first n powers of the matrix L and by the coefficients of its characteristic polynomial $\det(L - \lambda \mathbb{1})$.

In the paper [1], a set of $n-1$ new independent constants of motions are found, obviously not in involution with the functions λ_i. The basic step is to construct the so called *Moser coordinates* (r_i, λ_i) introduced in [3] using the *Weyl function*

$$
f(\lambda) = \frac{\Delta_{n-1}}{\Delta_n} \, ,
$$

where Δ_k is the subdeterminant obtained by cancelling the last $n - k$ rows and columns of the matrix $(\lambda \mathbb{1} - L)$. The Weyl function $f(\lambda)$ has n simple poles (coinciding with the eigenvalues λ_i of L) with positive residua such that $\sum_i \mathrm{res}_{\lambda_i} f(\lambda) = 1$. The positive functions r_i are related to the residua through the formula

$$
f(\lambda) = \frac{1}{K^2} \sum_{i=1}^{n} \frac{r_i^2}{\lambda - \lambda_i} \, , \quad K = \mathrm{e}^{\frac{1}{2}x_n} \tag{3}
$$

implying

$$
r_i^2 = K^2 \mathrm{res}_{\lambda_i} f(\lambda) \, . \tag{4}
$$

The map from coordinates (x_i, p_i) to coordinates (λ_i, r_i), with $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $r_i > 0$, is one to one and can be formally inverted. Indeed, $\sum_i \mathrm{res}_{\lambda_i} f(\lambda) = 1$ implies $K^2 = \sum_i r_i^2$ and $f(\lambda)$ admits a continued fraction expansion that goes back to Stieltjes [2] and plays a crucial role in [3]:

$$
f(\lambda) = \cfrac{2}{2\lambda + p_n - \cfrac{\mathrm{e}^{x_{n-1}-x_n}}{2\lambda + p_{n-1} - \cfrac{\mathrm{e}^{x_{n-2}-x_{n-1}}}{\ddots \cfrac{}{2\lambda + p_2 - \frac{\mathrm{e}^{x_1-x_2}}{2\lambda + p_1}}}}} \, . \tag{5}
$$

By comparing the expressions (5) and (3) one finds that p_i and e^{x_i} are rational functions of the coordinates (λ_i, r_i).

Moser coordinates greatly simplify equations (1): in fact, by differentiating the expression (4) and using the identity

$$
-\frac{1}{2}\mathrm{res}_{\lambda_i} \dot{f}(\lambda) = (\lambda_i + \frac{1}{2}p_n) \, \mathrm{res}_{\lambda_i} f(\lambda)
$$

2

proved in [3], one obtains

$$r_i \dot{r}_i = \frac{\dot{K}}{K} r_i^2 + \frac{1}{2} K^2 \operatorname{res}_{\lambda_i} \dot{f}(\lambda)$$

$$= \left[\frac{\dot{K}}{K} - \frac{1}{2} p_n - \lambda_i \right] r_i^2 .$$

The equations (1), written in Moser coordinates, then become

$$\begin{cases} \dot{\lambda}_i &= 0 \\ \dot{r}_i &= -r_i \lambda_i \end{cases} \tag{6}$$

because $\dot{K} = \frac{1}{2} K p_n$ and the eigenvalues λ_i are constant of the motion.

In the paper [1] it is shown, by direct computation, that for $k = 1, \ldots, n-1$ the functions

$$H_k = \left(\frac{r_k}{r_{k+1}} \right)^2 \exp \left(2 \frac{\lambda_k - \lambda_{k+1}}{\sum\limits_{i=1}^{n} \lambda_i} \sum_{i=1}^{n} \ln r_i \right) \tag{7}$$

are first integrals, and moreover the function H_k are independent both mutually and with respect to the λ_i. Hence the Toda lattice is a superintegrable system.

2 A set of simpler first integrals

The functions H_k given by (7) are pretty complicated. Furthermore, the fact that they are first integrals depends only on the very special form (6) of the system's equations. Indeed, as already noted in [1], by setting $\rho_i = \ln r_i$ these equations become

$$\begin{cases} \dot{\lambda}_i &= 0 \\ \dot{\rho}_i &= -\lambda_i \end{cases} \tag{8}$$

and it holds $\{\lambda_i, \lambda_j\} = \{\rho_i, \rho_j\} = 0$, $\{\lambda_i, \rho_j\} = \delta_{ij}$, i.e. the coordinates (ρ_i, λ_i) can be regarded as action-angles coordinates for the Toda lattice.

It is then very natural to look for a simpler form for the constants of motion, and try to generalize the argument to any system in action-angle coordinates. Using the coordinates (ρ_i, λ_i) one gets

$$H_k = \exp 2 \left(\rho_k - \rho_{k+1} \right) \exp 2 \left(\frac{\lambda_k - \lambda_{k+1}}{\sum\limits_{i=1}^{n} \lambda_i} \sum_{i=1}^{n} \rho_i \right)$$

$$= \exp 2 \frac{(\rho_k - \rho_{k+1}) \sum\limits_{i=1}^{n} \lambda_i + (\lambda_k - \lambda_{k+1}) \sum\limits_{i=1}^{n} \rho_i}{\sum\limits_{i=1}^{n} \lambda_i}$$

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and therefore, since $\sum_{i=1}^{n} \lambda_i$ is again a first integral, also the functions

$$\widetilde{H}_k = (\rho_k - \rho_{k+1})\sum_{i=1}^{n} \lambda_i + (\lambda_k - \lambda_{k+1})\sum_{i=1}^{n} \rho_i \qquad (9)$$

are constants of motion.

This is a particular case of a general situation:

Proposition 1 *If the phase space of an Hamiltonian system admits a set of coordinates (θ_i, I_i) such that the Hamilton equations take the form*

$$\left\{ \begin{array}{rcl} \dot{I}_i & = & 0 \\ \dot{\theta}_i & = & \omega_i(I_k) \end{array} \right. \qquad (10)$$

then either the functions I_k or the functions

$$K_k = \theta_k \sum_{i=1}^{n} \omega_i - \omega_k \sum_{i=1}^{n} \theta_i$$

are first integrals, for $k = 1, \ldots, n$.

Moreover if an index k_* exists, such that in an open dense subset of the phase space $\omega_{k_*} \neq 0$ and $\sum_i \omega_i \neq 0$, then the number of functions K_k functionally independent between themselves and with the functions I_k is exactly $2n - 1$.

Proof: Differentiating and using the Hamilton equations one obtains

$$\dot{K}_k = \omega_k \sum_{i=1}^{n} \omega_i - \omega_k \sum_{i=1}^{n} \omega_i = 0 \,.$$

At most $2n - 1$ of the function K_k are independent because

$$\sum_{k=1}^{n} K_k = \sum_{k=1}^{n} \theta_k \sum_{i=1}^{n} \omega_i - \sum_{k=1}^{n} \omega_k \sum_{i=1}^{n} \theta_i = 0$$

One can always suppose that the non-vanishing ω_{k_*} is ω_n then, setting

$$\begin{array}{rcl} \eta & = & \mathrm{d}I_1 \wedge \ldots \wedge \mathrm{d}I_n \\[2mm] \mathrm{d}\Theta & = & \displaystyle\sum_{i=1}^{n} \mathrm{d}\theta_i \\[2mm] \Omega & = & \displaystyle\sum_{i=1}^{n} \omega_i \end{array}$$

because $\eta \wedge \mathrm{d}\omega_j = 0$ one has

$$\eta \wedge \mathrm{d}K_2 \wedge \ldots \wedge \mathrm{d}K_n = \eta \wedge (\Omega \mathrm{d}\theta_2 - \omega_2 \mathrm{d}\Theta) \wedge \ldots \wedge (\Omega \mathrm{d}\theta_n - \omega_n \mathrm{d}\Theta)$$
$$= \Omega^{n-2}\eta \wedge \big[\Omega \mathrm{d}\theta_2 \wedge \ldots \wedge \mathrm{d}\theta_n -$$
$$\omega_2 \mathrm{d}\Theta \wedge \mathrm{d}\theta_3 \wedge \ldots \wedge \mathrm{d}\theta_n - \ldots - \omega_n \mathrm{d}\theta_2 \wedge \ldots \wedge \mathrm{d}\theta_{n-1} \wedge \mathrm{d}\Theta\big]$$

Hence the coefficient of $\eta \wedge \mathrm{d}\theta_1 \wedge \ldots \wedge \mathrm{d}\theta_{n-1}$ in $\eta \wedge \mathrm{d}K_2 \wedge \ldots \wedge \mathrm{d}K_n$ is $(-1)^{n-1}\Omega^{n-2}\omega_n$; this means that in the open dense subset of the phase space, where $\Omega \neq 0$ and $\omega_n \neq 0$, the differentials of the function $K_2 \ldots K_n$ and I_k are linearly independent, and therefore the corresponding functions are functionally independent. $\qquad\square$

Corollary 1 *If the Hamilton equations are given by (8), then the functions $K_k - K_{k+1}$ take the form (9) and hence they are $2n - 1$ constants of motion functionally independent between themselves and with the functions λ_i.*

Remark 1 *Another equivalent set of first integrals for the Hamilton equations (10) are the "generalized angular momenta"*

$$J_{ij} = \theta_i \omega_j - \omega_i \theta_j$$

that are simpler although their functional dependence is more cumbersome.

3   Final remarks

The superintegrability of the non-periodic Toda lattice is a special case of the superintegrability of a wider class of Hamiltonian systems. The key point is the existence in the phase space of the non-periodic Toda lattice of a global set of coordinates (the well known Moser coordinates [3]) of action-angles type. The extra constants of the motion found in [1] for the non-periodic Toda lattice are indeed functional combinations of simpler first integrals, that are defined for all systems admitting global action-angles coordinates.

References

[1] M. Agrotis, P. A. Damianou, C. Sophocleus, *The Toda lattice is super-integrable*, math-ph/0507051.

[2] F. R. Gantmacher, M. G. Krein, *Oscillation matrices and kernels and small vibrations of mechanical systems*, revised edition, AMS Chelsea Publishing 2002.

[3] J. Moser, *Finitely many mass points on the line under the influence of an exponential potential — An integrable system*, Lect. Notes Phys. **38** (1976), 97–101.